Approximation

GeoComput & ML

27 Apr. 2021

Interpolation

Definition

obtaining some function such that their values are identical to the given data

Definition

for given data

$$(t_i, y_i), \qquad i = 1, \ldots, m$$

we seek a function such that

$$\phi(t_i) = y_i, \qquad i = 1, \ldots, m$$

Motivation

Motivation

finite $\Leftrightarrow$ infinite

discrete $\Leftrightarrow$ continuous

Categorisation

- polynomial
- trigonometric
- piecewise

Polynomial Interpolation

Let $f(x)$ be the unknown function generating the data. We approximate $f(x)$ using a n degree polynomial $\phi_n(x) = \sum_{i=0}^{n} a_i\, x_i^i$ such that

$$f(x_i) = \phi_n(x_i), \qquad i = 0, \ldots, n$$

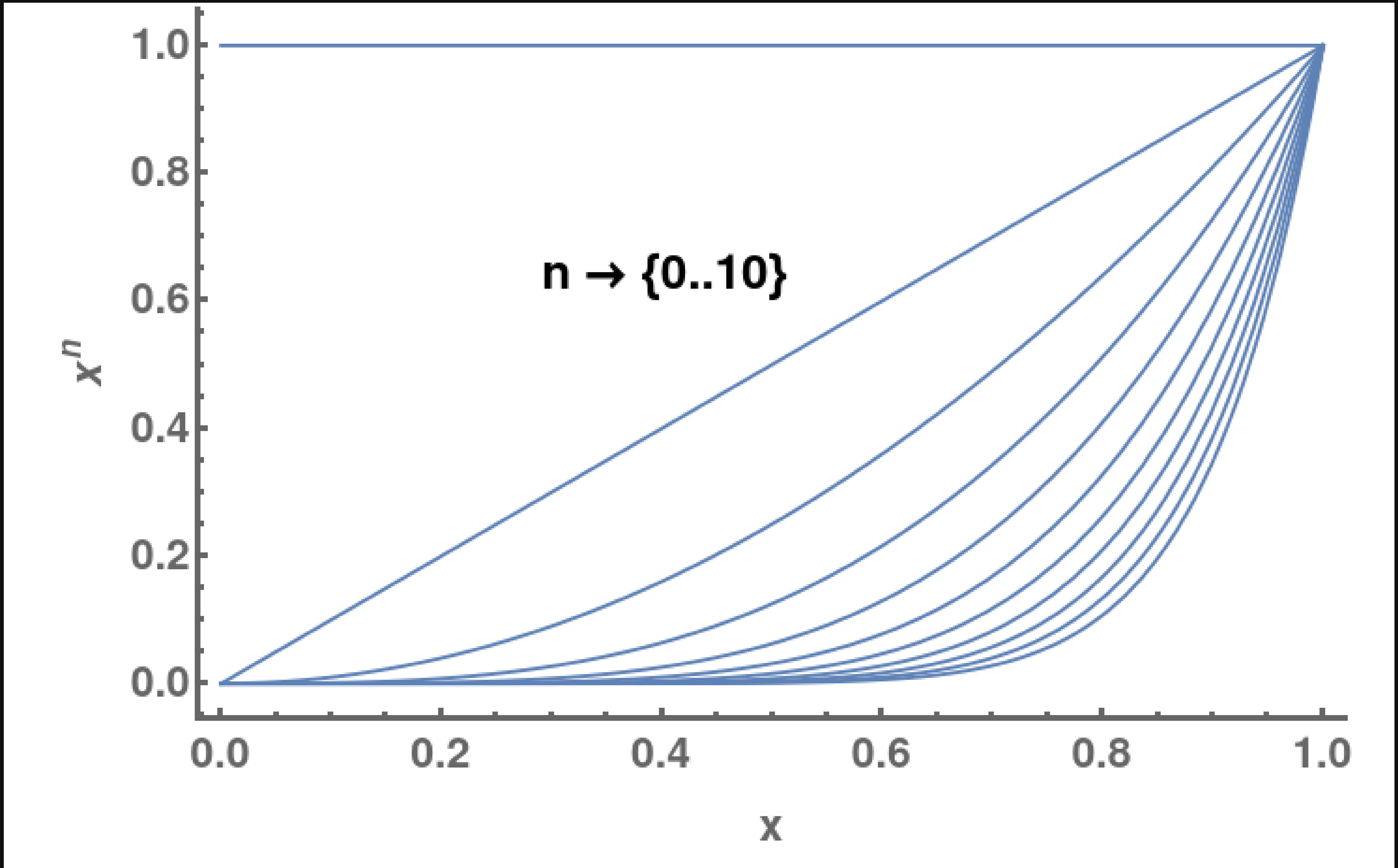
Polynomial Interpolation

that is $f(x_i) = \sum_i^n a_i \, x_i^i$

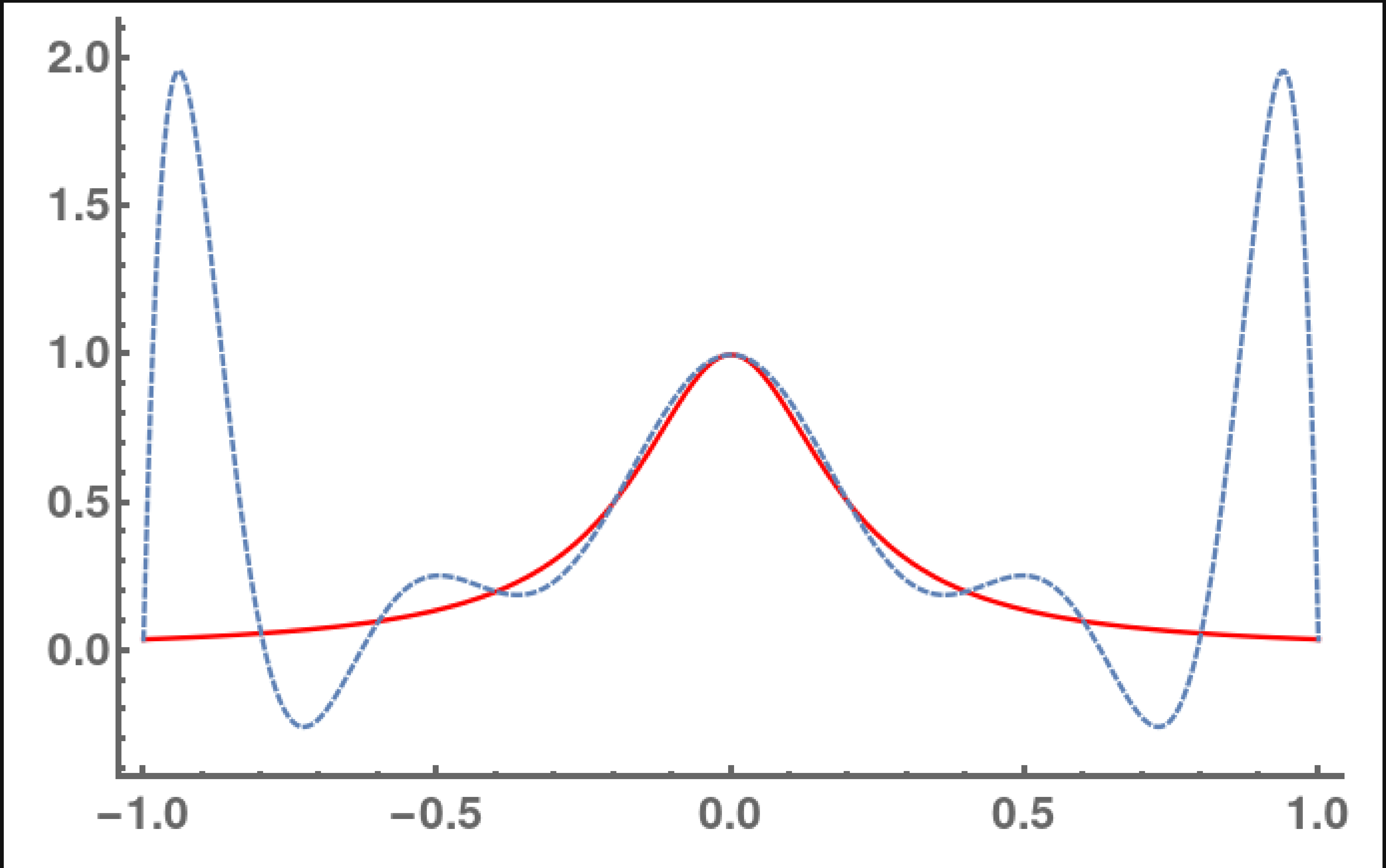
$(n-1)$ linear equations with coefficient determinant

$$\begin{vmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_0^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{vmatrix}$$

Polynomial Interpolation



Polynomial Interpolation



Piecewise Interpolation

Cubic Spline

Generally speaking, a spline is a polynomial of degree k with $k-1$ times continuous differentiabilities.

Cubic Spline

Let $f(x)$ be a function defined in the domain $a \leq x \leq b$. We partition the function into subintervals $a \leq x_0 < x_1 \ldots < x_n \leq b$

We aim to find a cubic function $s_{3,i}(x)$ such that

$$s_{3,i}(x_i) = f(x_i), \quad i = 0, \ldots, n-1$$

Cubic Spline

in each subinterval $[x_{i-1}, x_i]$, cubic spline $s_{3,i-1}(x_{i-1})$ must meet :

1. $s_{3,i-1}(x_{i-1}) = f(x_{i-1})$ and $s_{3,i}(x_i) = f(x_i)$
2. $s_{3,i}(x_i) = s_{3,i+1}(x_i)$
3. $s'_{3,i}(x_i) = s'_{3,i+1}(x_i)$
4. $s''_{3,i}(x_i) = s''_{3,i+1}(x_i)$

Cubic Spline

Question :

A cubic spline polynomial has $4(n-1)$ parameters to be determined. How many parameters can be fixed based on the previous constrains?

Hermite cubic spline

Hermite condition

$$H_{3,i-1}(x_{i-1}) = f(x_{i-1}), \quad H_{3,i}(x_i) = f(x_i)$$

$$H'_{3,i-1}(x_{i-1}) = f'(x_{i-1}), \quad H'_{3,i}(x_i) = f'(x_i)$$

Akima

Given a set of knot points (x_i, y_i) with x_i strictly increasing, Akima spline go through all the points and determine the slope for each point as a weighted average of the scants of two points before and after.

$$\boldsymbol{s}_i = \frac{|m_{i+1} - m_i| m_{i-1} + |m_{i-1} - m_{i-2}| m_i}{|m_{i+1} - m_i| + |m_{i-1} - m_{i-2}|}$$

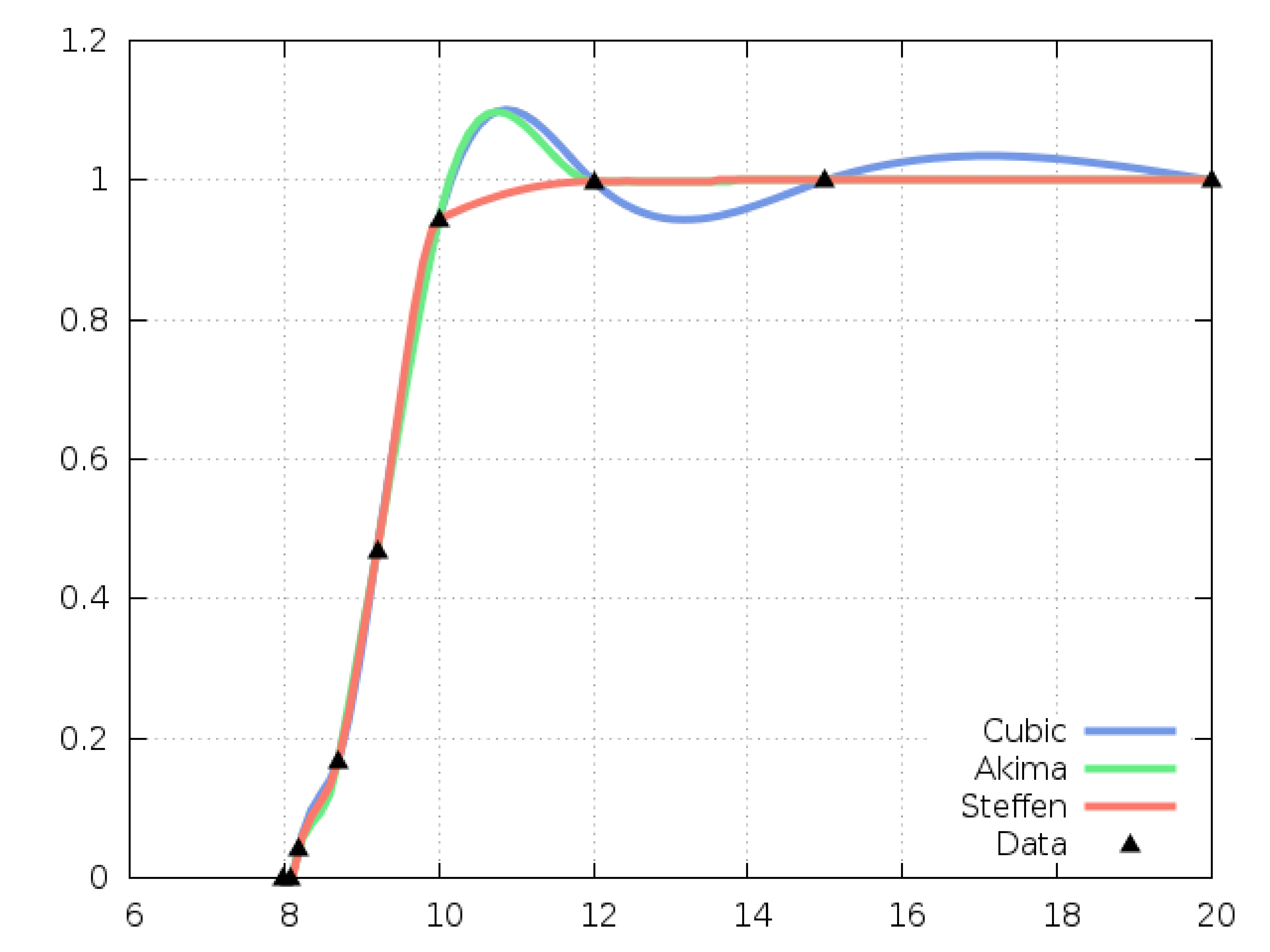
Steffen

estimate the slope of internal points through a unique parabola determined by three neighbouring points to ensure the monotonic behaviour of interpolation

$$p_i = \frac{s_{i-1}h_i + s_i h_{i-1}}{h_{i-1} + h_i}$$

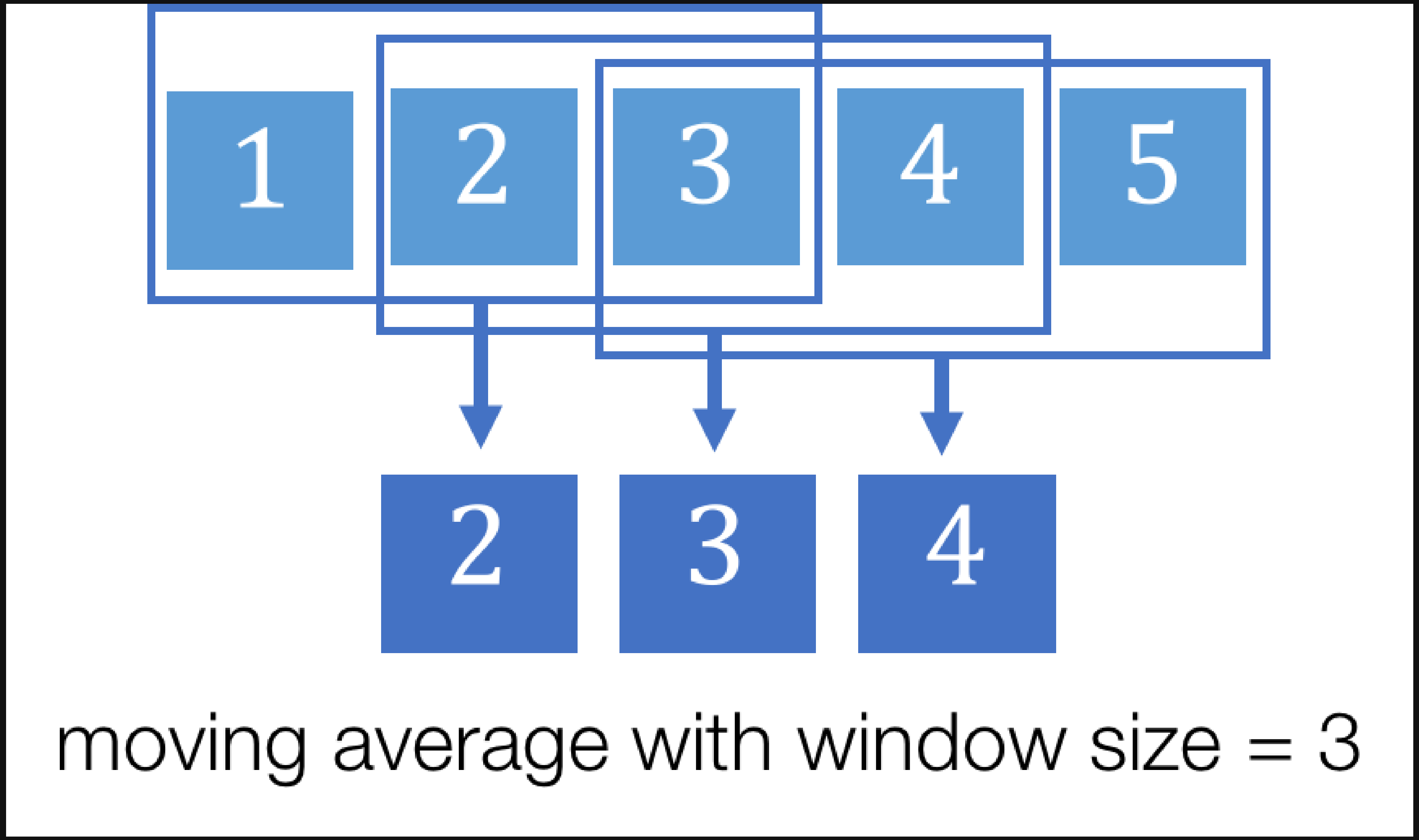
where $h_i = x_{i+1} - x_i$ and $s_i = \dfrac{y_{i+1} - y_i}{x_{i+1} - x_i}$

Comparison



Smoothing

Moving window



$$x_i^* = \frac{1}{2m+1} \sum_{j=-m}^{m} x_{i+j}$$

Salvitsky-Golay filtering

regression fitting

$$x^i_j = \sum_{l=0}^{k-1} a_l j^l, \quad j \in [-m, m], i \in [1, n]$$

$$\boldsymbol{x} = \boldsymbol{M}\,\boldsymbol{a}$$
$$\boldsymbol{a} = (\boldsymbol{M}^T\boldsymbol{M})\boldsymbol{M}\,\boldsymbol{x}$$
$$\hat{\boldsymbol{x}} = \boldsymbol{M}(\boldsymbol{M}^T\boldsymbol{M})\boldsymbol{M}\,\boldsymbol{x}$$

Fourier Transform

Fourier Series

representation of a function $f(x)$ in terms of a set of trigonometric functions

$$cos(n\,x), \quad n = 0, 1, 2, 3, \ldots$$
$$sin(n\,x), \quad n = 1, 2, 3, \ldots$$

Fourier Series

- orthogonality

$$\int_{-\pi}^{\pi} cos\,mx\;cos\,nx\;dx = 0, \quad m \neq n$$
$$\int_{-\pi}^{\pi} sin\,mx\;sin\,nx\;dx = 0, \quad m \neq n$$
$$\int_{-\pi}^{\pi} sin\,mx\;cos\,nx\;dx = 0, \quad any\;m, n$$

$$\int_{-\pi}^{\pi} cos\,nx\;cos\,nx\;dx = 2\pi\;or\;\pi, \quad if\;n = 0\;or\;n > 0$$
$$\int_{-\pi}^{\pi} sin\,nx\;sin\,nx\;dx = \pi, \quad if\;n = 0$$

Fourier Series

$$\int_{-\pi}^{\pi} cos\, mx\ cos\, nx\ dx = \frac{1}{2}\int_{-\pi}^{\pi}(cos(m+n)x + cos(m-n)x)\, dx$$

$$= \frac{1}{2}\left[\frac{sin(m+n)x}{m+n} + \frac{sin(m-n)x}{m-n}\right]_{-\pi}^{\pi}$$

$$= 0$$

Fourier Series

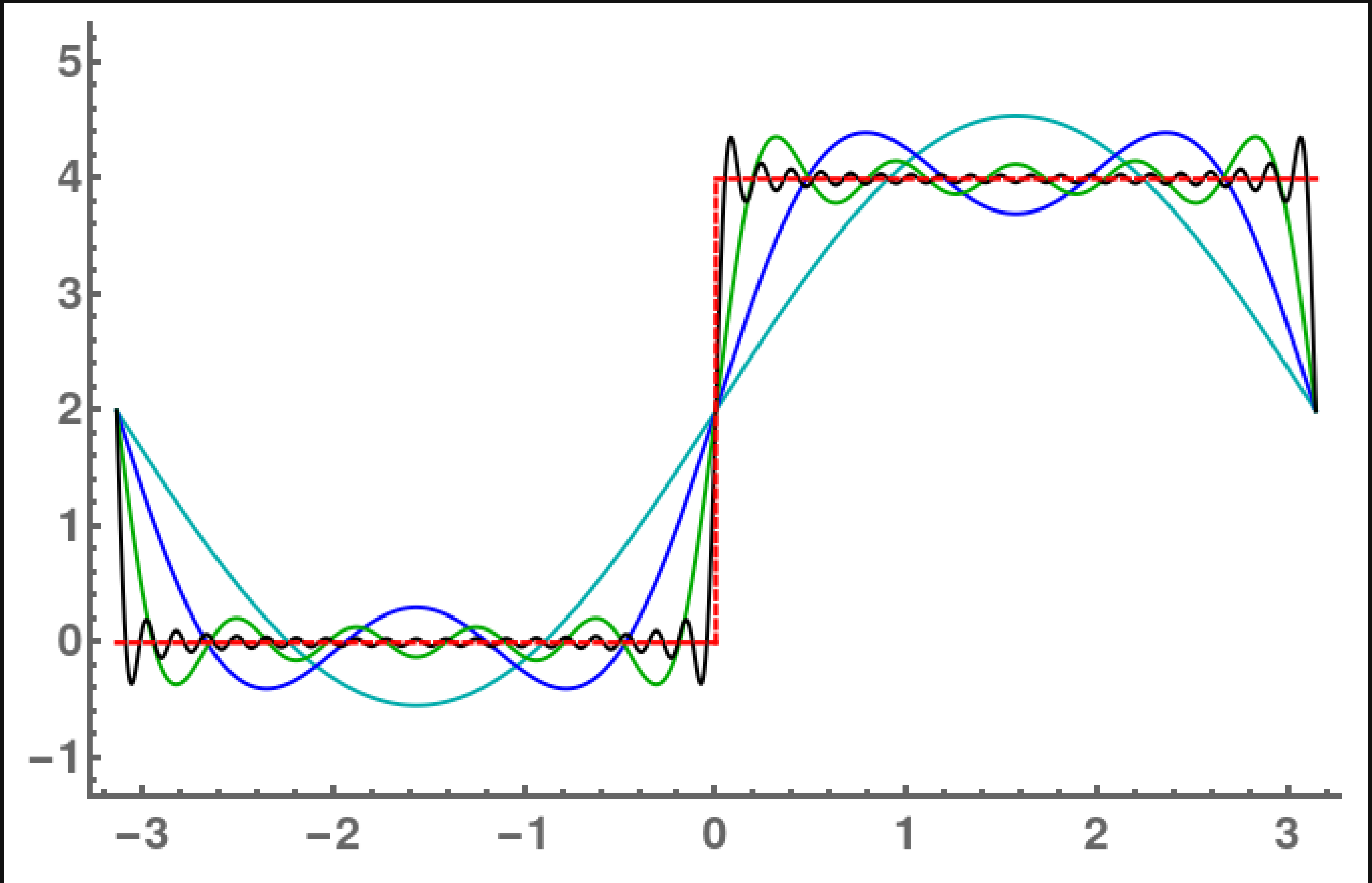
Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cos\, nx + b_n sin\, nx)$$

Fourier coefficients

$$\begin{cases} a_n = \int_{-\pi}^{\pi} f(x) cos(nx)\, dx \\ b_n = \int_{-\pi}^{\pi} f(x) sin(nx)\, dx \end{cases}$$

Fourier Series



Fourier Series

FS in complex exponential form

$$cos\theta = (e^{i\theta} + e^{-i\theta})/2; \quad sin\theta = (e^{i\theta} - e^{-i\theta})/(2i)$$

$$FS\, f = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$$

where $c_n = \dfrac{1}{2l}\displaystyle\int_{-l}^{l} f(x) e^{-in\pi x/l} dx$

Fourier Integral

$$f(x) = \int_0^{\infty} (a(\omega)cos(\omega x) + b(\omega)sin(\omega x))d\omega$$

$$\begin{cases} a(\omega) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(x)cos(\omega x)dx \\ b(\omega) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(x)sin(\omega x)dx \end{cases}$$

Fourier Transform

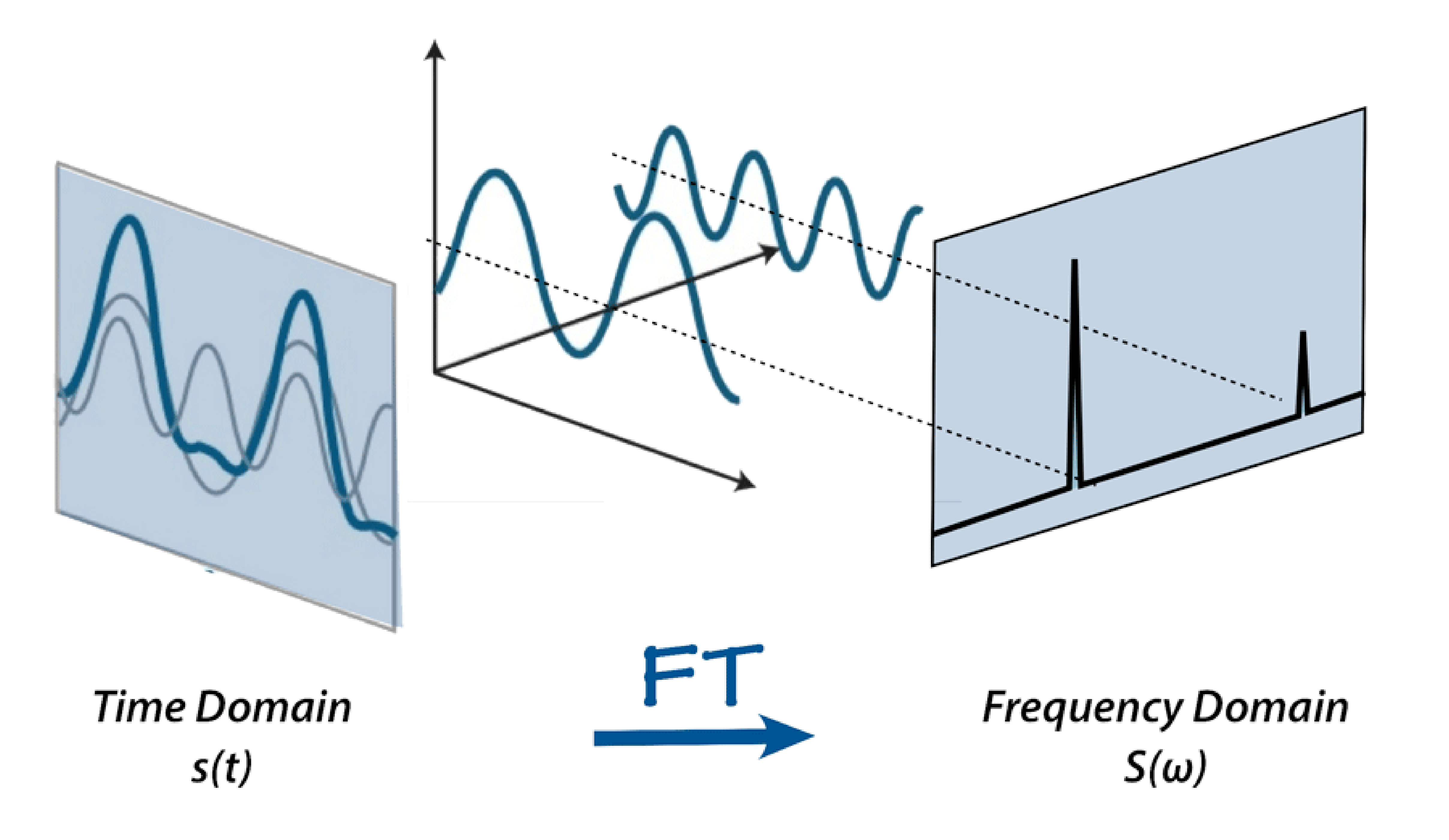
$$
\begin{aligned}
f(x) &= \frac{1}{\pi}\int_{0}^{\infty}\left\{\int_{-\infty}^{\infty} f(\xi)[cos(\omega\xi)cos(\omega x) + sin(\omega\xi)sin(\omega x)]d\xi\right\}d\omega \\
&= \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(\xi)cos\omega(\xi - x)d\xi d\omega \\
&= \frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(\xi)e^{i\omega(\xi-x)}d\xi d\omega + \frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(\xi)e^{-i\omega(\xi-x)}d\xi d\omega \\
&= \frac{1}{2\pi}\int_{0}^{-\infty}\int_{-\infty}^{\infty} f(\xi)e^{-i\omega(\xi-x)}d\xi(-d\omega) + \frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(\xi)e^{-i\omega(\xi-x)}d\xi d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{0}\int_{-\infty}^{\infty} f(\xi)e^{-i\omega(\xi-x)}d\xi d\omega + \frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(\xi)e^{-i\omega(\xi-x)}d\xi d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\xi)e^{-i\omega\xi}d\xi\right]e^{i\omega x}d\omega
\end{aligned}
$$

Fourier Transform

$$F\{f(x)\} = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

$$F^{-1}\{\hat{f}(\omega)\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega$$

Fourier Transform

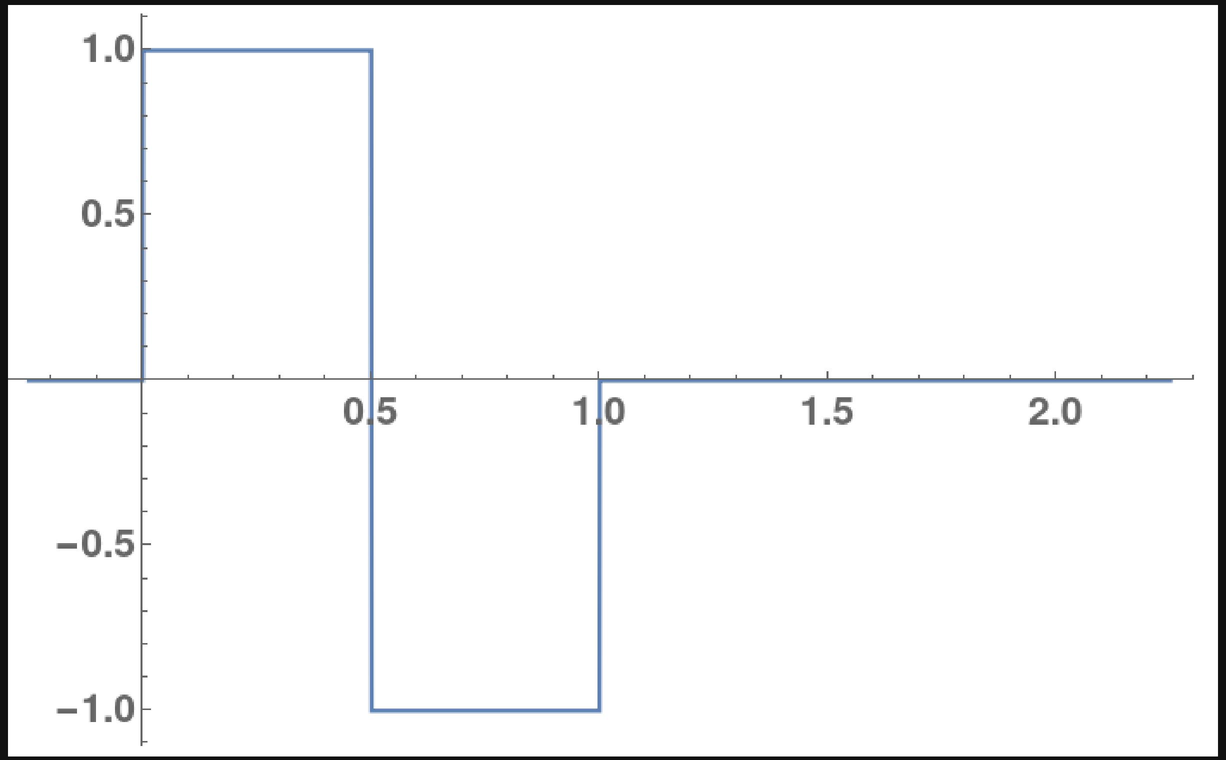


Wavelets

Haar Wavelets

Mother Haar Wavelet

$$\psi(x) = \begin{cases} 1 & 0 \leq x \leq 1/2 \\ -1 & 1/2 \leq x \leq 1 \\ 0 & otherwise \end{cases}$$



Haar Wavelets

Haar Wavelets Family

$$\psi_{j,k}(x) = 2^{j/2}(\psi(2^j x - k))$$

where $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$

Haar Wavelets

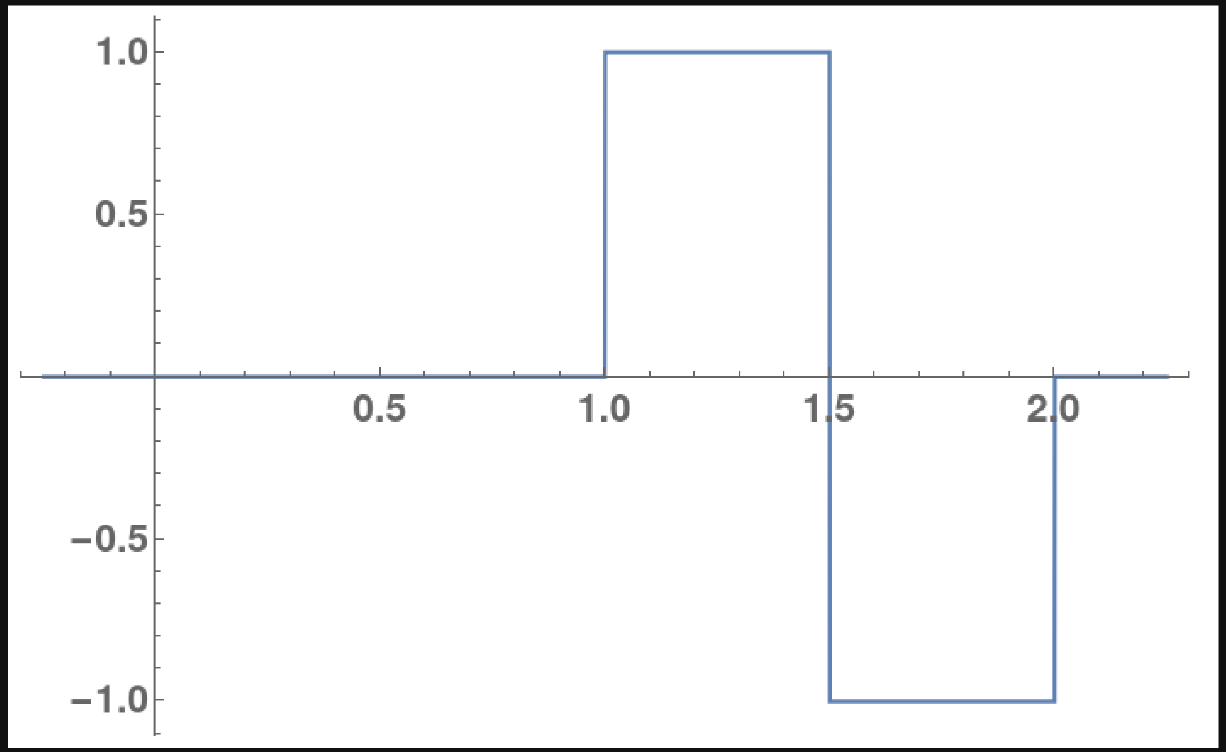
def. supp $f = \{x \in X \mid f(x) \neq 0\}$

supp $\psi_{j,k}(X) = [2^{-j}k, 2^{-j+1}k)$

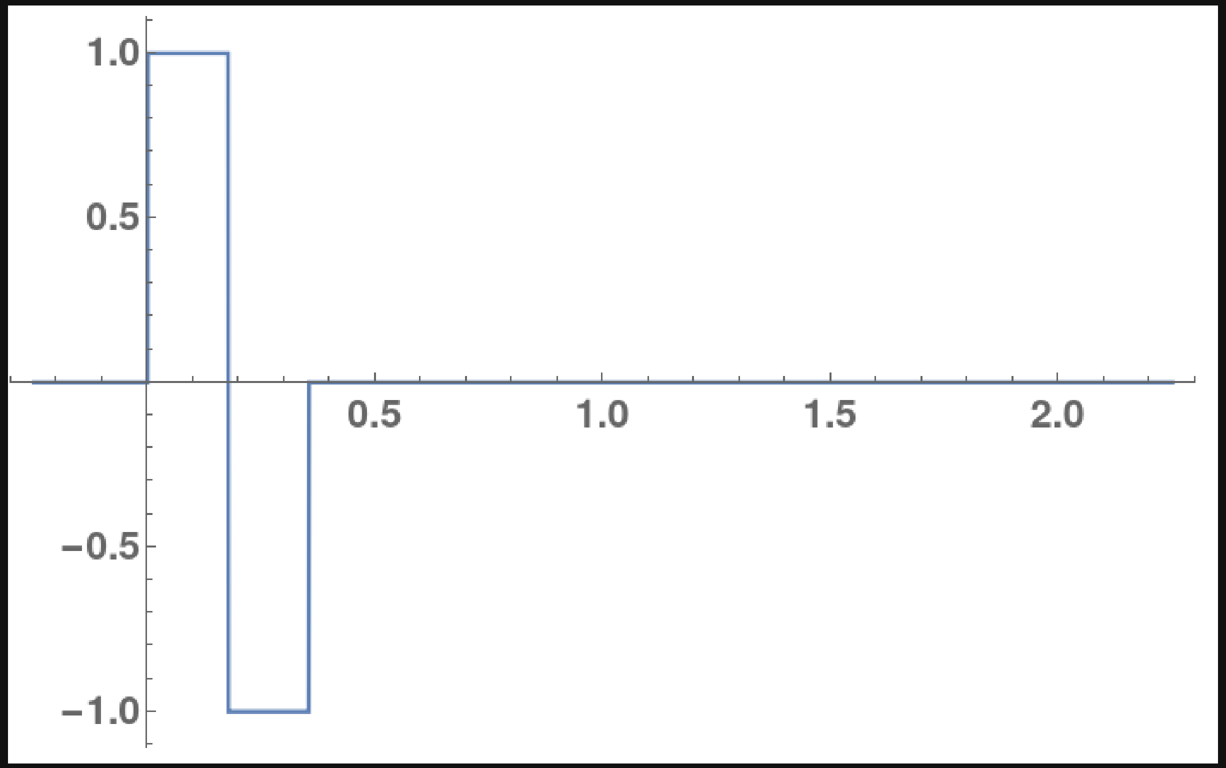
- Either the dyadic intervals non-overlapping or one contained in another
- If in containment, then one is either contained to the left or right part of the interval

Haar Wavelets

$\psi_{0,1}$



$\psi_{1,0}$



Haar Wavelets

Orthogonality

$$\left.\begin{aligned}\langle\psi_{j,k},\psi_{j',k'}\rangle &= \int_{-\infty}^{\infty} 2^{j/2}\psi(2^j x-k)\,2^{j'/2}\psi(2^{j'}x-k')dx \\ u &= 2^j x - k\end{aligned}\right\}\Rightarrow$$

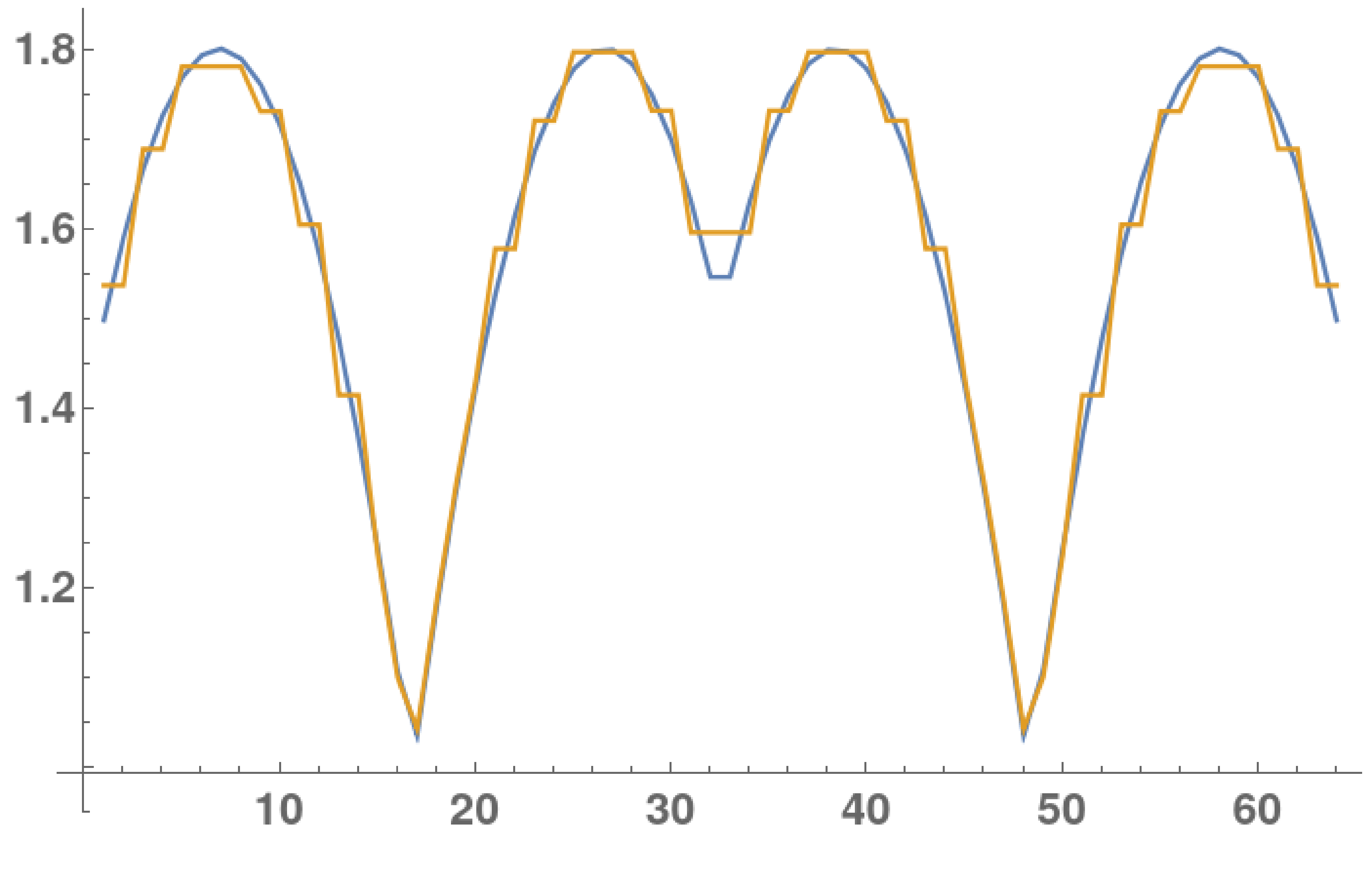
$$\langle\psi_{j,k},\psi_{j',k'}\rangle = \int_{-\infty}^{\infty} 2^{(j'-j)/2}\psi(u)\psi(2^{(j'-j)/2}u - 2^{(j'-j)/2}k + k')du$$

Haar Wavelets

$$f(x) = \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} d_{j,k} \psi_{j,k}(x)$$

where $d_{j,k} = \langle f(x), \psi_{j,k}(x) \rangle$, wavelet coefficients

Haar Wavelets

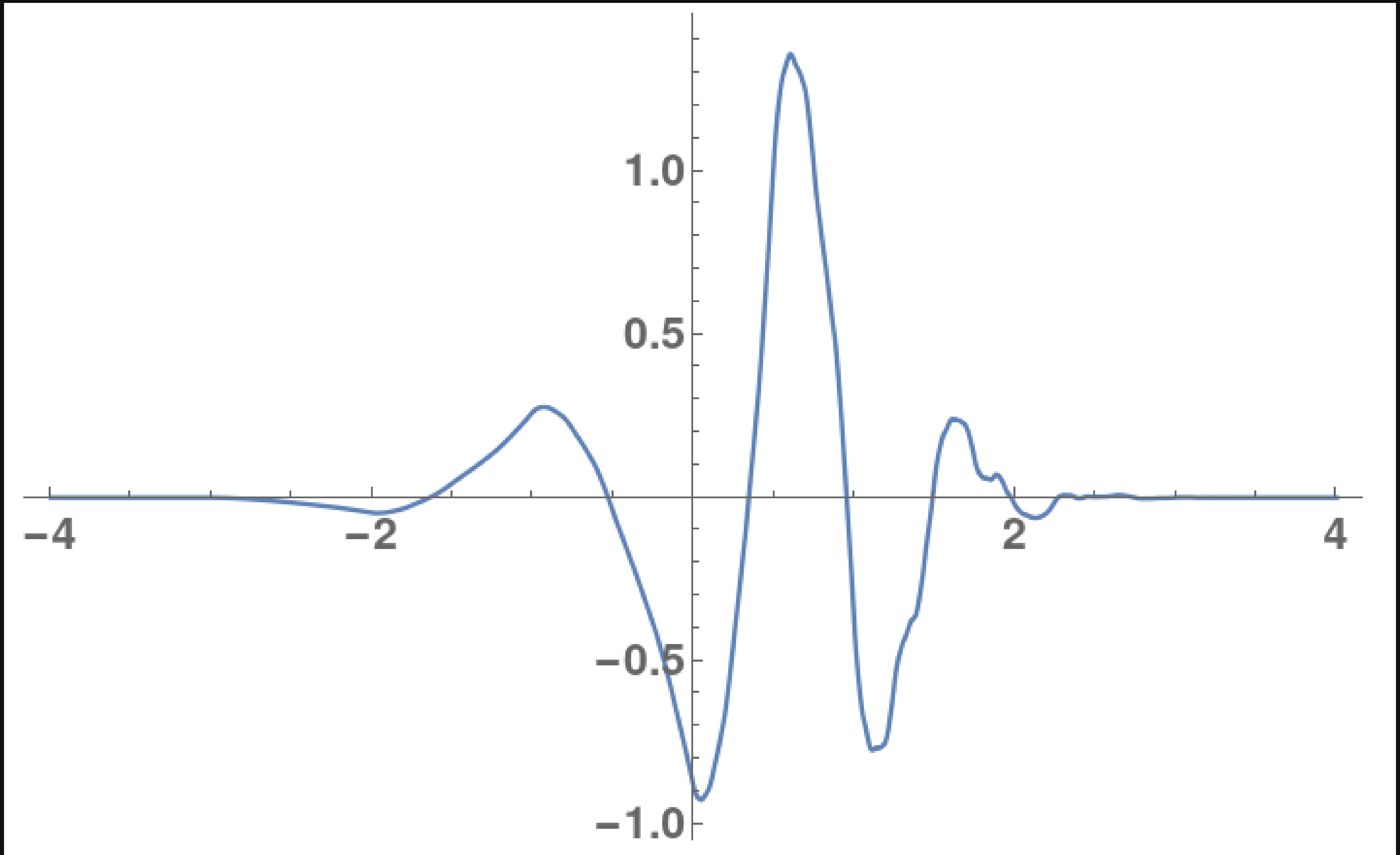


Daubechies Wavelets

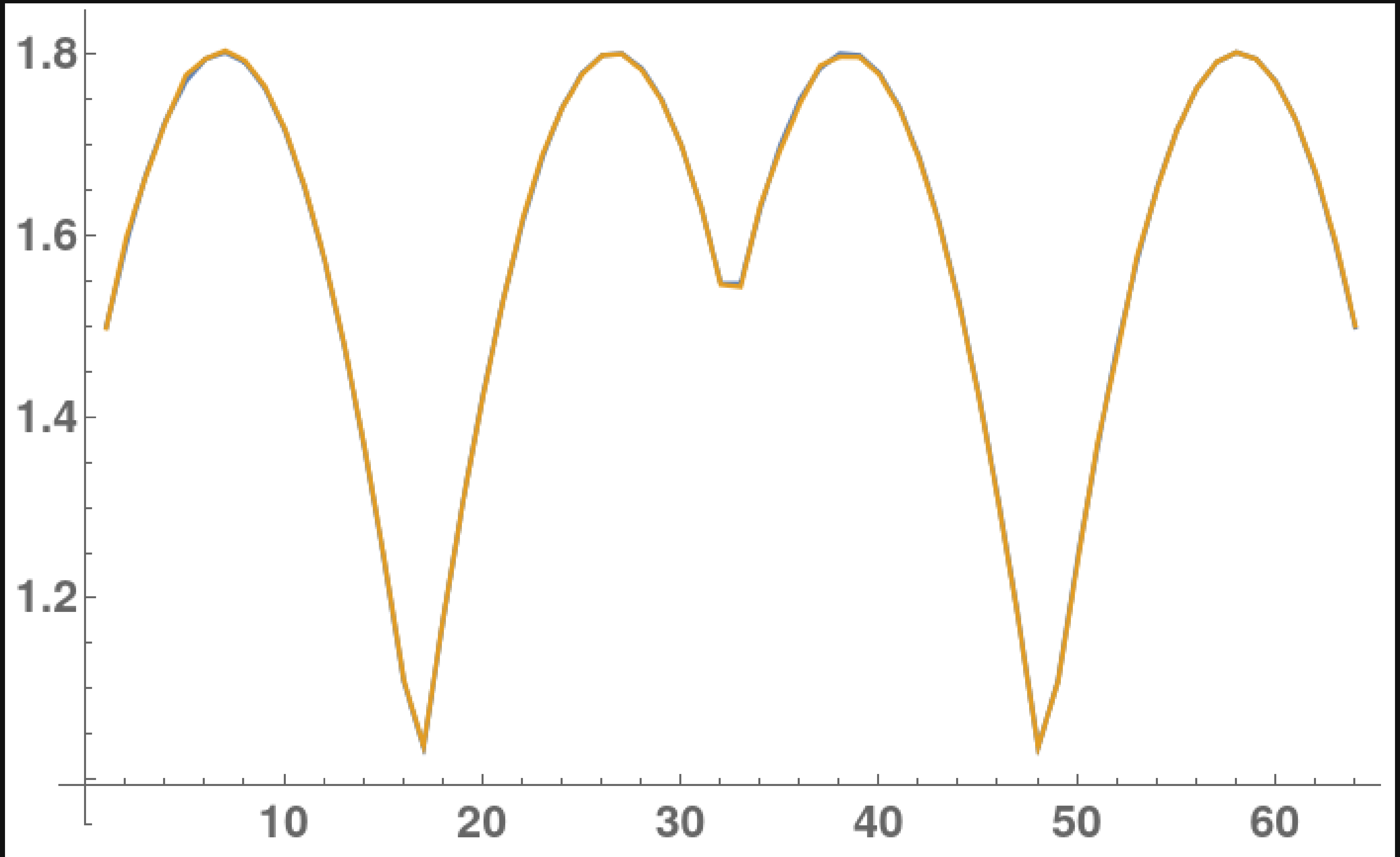
$$\begin{cases} \phi(x) = \displaystyle\sum_{k=0}^{2N-1} a_k \phi(2x + k) \\ \\ \psi(x) = \displaystyle\sum_{k=0}^{2N-1} (-1)^{k-1} a_k \phi(2x + k - 1) \end{cases}$$

$$\begin{cases} a_0 = \dfrac{1}{4}(1 + \sqrt{3}); & a_1 = \dfrac{1}{4}(3 + \sqrt{3}) \\ a_2 = \dfrac{1}{4}(3 - \sqrt{3}); & a_3 = \dfrac{1}{4}(1 - \sqrt{3}) \end{cases}$$

Daubechies Wavelets



Daubechies Wavelets



Daubechies Wavelets

If $f \in \mathbb{R}$ and $m \in \mathbb{Z}_{\geq 0}$, then

$$\int_{-\infty}^{\infty} x^m \, f(x) dx$$

if it exists, it is called the moment of f of order m.

f has the **vanishing moments** to order M if the moment of f of order $m = 0, 1, \ldots, M$ are all 0.

Daubechies Wavelets

Wavelets choice

- moments and support
- feature of the target function f
- customisation

Acknowledgement

Thanks for Your Attention

References

- H. Akima, JACM. (1970) 17, 589
- M. Steffen, Astron. Astrophys. (1990) 239, 443
- M. Heath, Scientific Computing An Introductory Survey, 2018
- S. Ray, Numerical analysis with algorithms and programming, 2016
- P. Wojtaszczyk, A mathematical introduction to wavelets, 1997
- R. Ogden, Essential wavelets for statistical applications and data analysis, 1997
- G. Bachman, *et. al.* Fourier and wavelet analysis, 2000
- P. van Fleet, Discrete wavelet transformations _ an elementary approach with applications, 2008
- D. Sundararajan, Discrete Wavelet Transform_ A Signal Processing Approach
- J. Winefordner, Chemometrics From Basics to Wavelets, 2004
- E. Steiner, The Chemistry Maths Book, 1996
- M. Greenberg, Advanced Engineering Mathematics, 2004
- https://www.gnu.org/software/gsl/doc/html/interp.html
- https://dawn.cs.stanford.edu/2017/08/07/asap/
- http://mri-q.com/fourier-transform-ft.html